2700 [May, 1918]. Proposed by the late ARTEMAS MARTIN.

In a factory 250 men are paid an average wage of \$15 each per week. The men are paid unequally, the wages being \$20, \$16, \$10, and \$8 per week, respectively, for different classes of work. How many are employed at each rate of pay?

NOTE.—I am told that this question was set in a Civil Service examination paper to be worked by arithmetic. 2,896 answers have been found. Are there any more?

SOLUTION BY H. S. UHLER, Yale University.

Let w, x, y, and z denote the number of men receiving \$20, \$16, \$10, and \$8 per week, respectively. The conditions of the problem lead at once to the following equations:

$$w + x + y + z = 250,$$

$$(20w + 16x + 10y + 8z)/250 = 15.$$

Elimination of z gives

$$6w + 4x + y = 875.$$

The last equation must be solved for positive integral values (zero included) of w, x, and y. This may be accomplished by assigning to w the values 0, 1, 2, 3, $\cdots$ and then discussing the number of possible solutions conditioned by the limited arithmetical progressions involving x, y, and z.

Since $y = 875 - 6w - 4x$ it is evident that the greatest value of w which will make y positive is 145. Let w' symbolize any number in the sequence 0, 1, 2, 3, $\cdots$, 144, 145. We may now imagine the following table filled out numerically:

w	x	y	z
w'	x'	$875 - 6w' - 4x'$	$5w' - 625 + 3x'$
w'	$x' - 1$	$875 - 6w' - 4x' + 4$	$5w' - 625 + 3x' - 3$
w'	$x' - 2$	$875 - 6w' - 4x' + 8$	$5w' - 625 + 3x' - 6$
.	. .	. . . .	. . . .

For a given value of w (w') the number of possible solutions is equal to the greatest number of rows that can be written in the above schematic table without introducing a negative value in one or more of the last three columns.

CASE 1.—*w' odd.* Let $w' = 2k + 1$, then

$$y = 1 + 4(217 - 3k - x'),$$

hence the greatest integral value of x' which will make y positive is $217 - 3k$. Accordingly, when w' is odd, the first or top row of the table will consist of the elements $2k + 1$, $217 - 3k$, 1, and $k + 31$ under the headings w, x, y, and z, respectively. Since the greatest value of w' is 145 the corresponding value of k equals 72, hence the least value of $217 - 3k$ is 1 so that all the elements of the above first row are positive.

Attention will now be directed to the last or bottom row of the table. When the total number of rows is $218 - 3k$ the elements of the last row will be $2k + 1$, 0, $869 - 12k$, and $10(k - 62)$, for the common differences of the second, third, and fourth columns are -1, 4, and -3, respectively. Therefore, as long as k does not fall below 62 the second column will limit the number of rows in the table. Under the conditions that w' be odd and $k \geqq 62$ the number of rows in a table equals $218 - 3k$ so that the number of solutions of the problem is the sum $2 + 5 + \cdots + 29 + 32 = 187$.

For values of k less than 62 the fourth column will limit the number or rows in the table, instead of the second column. Let k be of the form $3m + 2$, then the nth term of the fourth column may be written $z_n = 3(12 + m - n)$. This will vanish whenever $n = 12 + m$. Positive solutions for k, equal to $3m + 2$, and less than 62, are obtained when $m = 19, 18, \cdots$, or $k = 59, 56, \cdots$, or $w' = 119, 113, \cdots$. When $k = 3m + 3$ and $k = 3m + 4$ the $(12 + m)$th terms of the fourth column will be 1 and 2, respectively. On the other hand, the $(13 + m)$th terms will be -2 and -1, in the same order. Consequently, for odd values of w', corresponding to values of k less than 62, the tables may be collected in groups of three having the same number of rows or possible solutions. The number of rows is evidently $m + 12$. If we allow m to decrease from 19 to -1, or k from 61, $(3m + 4)$, to -1, $(3m + 2)$, we shall obtain not only

the number of possible solutions pertaining to values of w' from 123 to 1 but also one inadmissible solution belonging to $w' = -1$. Consequently, the number of possible solutions associated with odd values of w' and not included in the 187 given above is equal to $3(31 + 30 + \cdots + 12 + 11) - 11 = 1312$. Finally, the total number of possible solutions of the problem, when w' is odd, equals $187 + 1312 = 1499$.

CASE 2.—*w' even.* Let $w' = 2k$, then

$$y = 3 + 4(218 - 3k - x'),$$

hence $218 - 3k$ is the greatest integral value of x' that will make y positive. Accordingly, when w' is even, the first row of the table will comprise the constituents $2k$, $218 - 3k$, 3, and $29 + k$ under the headings w, x, y, and z, respectively. Since the greatest even value of w' is 144 the corresponding value of k equals 72, hence the least value of $218 - 3k$ is 2 so that all the elements of the above first row are positive.

When the total number of rows is $219 - 3k$ the elements of the last row will be $2k$, 0, $875 - 12k$, and $10k - 625$. Therefore, as long as k is not less than 63 the second column will limit the number of rows in the table. Under the present conditions the number of rows in a table equals $219 - 3k$ so that the number of solutions of the problem is the sum

$$3 + 6 + \cdots + 27 + 30 = 165.$$

For values of k less than 63 the fourth column will limit the number of rows in the table. Let k be of the form $3m + 1$, then the nth term of the fourth column may be written $z_n = 3(11 + m - n)$. This will vanish whenever $n = 11 + m$. Positive solutions for k, equal to $3m + 1$, and less than 63, are obtained when $m = 20, 19, \cdots$, or $k = 61, 58, \cdots$, or $w' = 122, 116, \cdots$. When $k = 3m + 2$ and $k = 3m + 3$ the $(11 + m)$th terms of the fourth column will be 1 and 2 respectively. On the other hand, the $(12 + m)$th terms will be -2 and -1, in the order named. Consequently, for even values of w', corresponding to values of k less than 63, the tables may be collected in groups of three having the same number of rows or possible solutions. The number of rows is obviously $m + 11$. If we allow m to decrease from 20 to 0, or k from 63, $(3m + 3)$, to 1, $(3m + 1)$, we shall include the case of $w' = 126$ which has already been disposed of and omit the number of solutions for $w' = 0$, namely 10. Consequently, the number of solutions not already accounted for equals $3(31 + 30 + \cdots + 12 + 11) - 31 + 10 = 1302$. Finally, the total number of possible solutions of the problem, when w' is even (or zero), equals $165 + 1302 = 1467$. Therefore, for all admissible values of w', the complete number of solutions is 2966. Hence 70 more solutions exist than the proposer states have already been found.

REMARK.—The number of solutions involving one or more zeros is 72.

N = number of solutions in the table.
N' = number of rows containing one or more zeros.
Only the first and last rows are given for each value of w.

w	x	y	z	N	N'	w	x	y	z	N	N'	w	x	y	z	N	N'
145	1	1	103	2	1	137	13	1	99	14	1	129	25	1	95	26	1
145	0	5	100			137	0	53	60			129	0	101	20		
144	2	3	101	3	1	136	14	3	97	15	1	128	26	3	93	27	1
144	0	11	95			136	0	59	55			128	0	107	15		
143	4	1	102	5	1	135	16	1	98	17	1	127	28	1	94	29	1
143	0	17	90			135	0	65	50			127	0	113	10		
142	5	3	100	6	1	134	17	3	96	18	1	126	29	3	92	30	1
142	0	23	85			134	0	71	45			126	0	119	5		
141	7	1	101	8	1	133	19	1	97	20	1	125	31	1	93	32	1
141	0	29	80			133	0	77	40			125	0	125	0		
140	8	3	99	9	1	132	20	3	95	21	1	124	32	3	91	31	0
140	0	35	75			132	0	83	35			124	2	123	1		
139	10	1	100	11	1	131	22	1	96	23	1	123	34	1	92	31	0
139	0	41	70			131	0	89	30			123	4	121	2		
138	11	3	98	12	1	130	23	3	94	24	1	122	35	3	90	31	1
138	0	47	65			130	0	95	25			122	5	123	0		

w	x	y	z	N	N'	w	x	y	z	N	N'	w	x	y	z	N	N'
121	37	1	91	31	0	96	74	3	77	26	0	71	112	1	66	23	1
121	7	121	1			96	49	103	2			71	90	89	0		
120	38	3	89	30	0	95	76	1	78	27	1	70	113	3	64	22	0
120	9	119	2			95	50	105	0			70	92	87	1		
119	40	1	90	31	1	94	77	3	76	26	0	69	115	1	65	22	0
119	10	121	0			94	52	103	1			69	94	85	2		
118	41	3	88	30	0	93	79	1	77	26	0	68	116	3	63	22	1
118	12	119	1			93	54	101	2			68	95	87	0		
117	43	1	89	30	0	92	80	3	75	26	1	67	118	1	64	22	0
117	14	117	2			92	55	103	0			67	97	85	1		
116	44	3	87	30	1	91	82	1	76	26	0	66	119	3	62	21	0
116	15	119	0			91	57	101	1			66	99	83	2		
115	46	1	88	30	0	90	83	3	74	25	0	65	121	1	63	22	1
115	17	117	1			90	59	99	2			65	100	85	0		
114	47	3	86	29	0	89	85	1	75	26	1	64	122	3	61	21	0
114	19	115	2			89	60	101	0			64	102	83	1		
113	49	1	87	30	1	88	86	3	73	25	0	63	124	1	62	21	0
113	20	117	0			88	62	99	1			63	104	81	2		
112	50	3	85	29	0	87	88	1	74	25	0	62	125	3	60	21	1
112	22	115	1			87	64	97	2			62	105	83	0		
111	52	1	86	29	0	86	89	3	72	25	1	61	127	1	61	21	0
111	24	113	2			86	65	99	0			61	107	81	1		
110	53	3	84	29	1	85	91	1	73	25	0	60	128	3	59	20	0
110	25	115	0			85	67	97	1			60	109	79	2		
109	55	1	85	29	0	84	92	3	71	24	0	59	130	1	60	21	1
109	27	113	1			84	69	95	2			59	110	81	0		
108	56	3	83	28	0	83	94	1	72	25	1	58	131	3	58	20	0
108	29	111	2			83	70	97	0			58	112	79	1		
107	58	1	84	29	1	82	95	3	70	24	0	57	133	1	59	20	0
107	30	113	0			82	72	95	1			57	114	77	2		
106	59	3	82	28	0	81	97	1	71	24	0	56	134	3	57	20	1
106	32	111	1			81	74	93	2			56	115	79	0		
105	61	1	83	28	0	80	98	3	69	24	1	55	136	1	58	20	0
105	34	109	2			80	75	95	0			55	117	77	1		
104	62	3	81	28	1	79	100	1	70	24	0	54	137	3	56	19	0
104	35	111	0			79	77	93	1			54	119	75	2		
103	64	1	82	28	0	78	101	3	68	23	0	53	139	1	57	20	1
103	37	109	1			78	79	91	2			53	120	77	0		
102	65	3	80	27	0	77	103	1	69	24	1	52	140	3	55	19	0
102	39	107	2			77	80	93	0			52	122	75	1		
101	67	1	81	28	1	76	104	3	67	23	0	51	142	1	56	19	0
101	40	109	0			76	82	91	1			51	124	73	2		
100	68	3	79	27	0	75	106	1	68	23	0	50	143	3	54	19	1
100	42	107	1			75	84	89	2			50	125	75	0		
99	70	1	80	27	0	74	107	3	66	23	1	49	145	1	55	19	0
99	44	105	2			74	85	91	0			49	127	73	1		
98	71	3	78	27	1	73	109	1	67	23	0	48	146	3	53	18	0
98	45	107	0			73	87	89	1			48	129	71	2		
97	73	1	79	27	0	72	110	3	65	22	0	47	148	1	54	19	1
97	47	105	1			72	89	87	2			47	130	73	0		

w	x	y	z	N	N'	w	x	y	z	N	N'	w	x	y	z	N	N'
46	149	3	52	18	0	30	173	3	44	15	0	14	197	3	36	13	1
46	132	71	1			30	159	59	2			14	185	51	0		
45	151	1	53	18	0	29	175	1	45	16	1	13	199	1	37	13	0
45	134	69	2			29	160	61	0			13	187	49	1		
44	152	3	51	18	1	28	176	3	43	15	0	12	200	3	35	12	0
44	135	71	0			28	162	59	1			12	189	47	2		
43	154	1	52	18	0	27	178	1	44	15	0	11	202	1	36	13	1
43	137	69	1			27	164	57	2			11	190	49	0		
42	155	3	50	17	0	26	179	3	42	15	1	10	203	3	34	12	0
42	139	67	2			26	165	59	0			10	192	47	1		
41	157	1	51	18	1	25	181	1	43	15	0	9	205	1	35	12	0
41	140	69	0			25	167	57	1			9	194	45	2		
40	158	3	49	17	0	24	182	3	41	14	0	8	206	3	33	12	1
40	142	67	1			24	169	55	2			8	195	47	0		
39	160	1	50	17	0	23	184	1	42	15	1	7	208	1	34	12	0
39	144	65	2			23	170	57	0			7	197	45	1		
38	161	3	48	17	1	22	185	3	40	14	0	6	209	3	32	11	0
38	145	67	0			22	172	55	1			6	199	43	2		
37	163	1	49	17	0	21	187	1	41	14	0	5	211	1	33	12	1
37	147	65	1			21	174	53	2			5	200	45	0		
36	164	3	47	16	0	20	188	3	39	14	1	4	212	3	31	11	0
36	149	63	2			20	175	55	0			4	202	43	1		
35	166	1	48	17	1	19	190	1	40	14	0	3	214	1	32	11	0
35	150	65	0			19	177	53	1			3	204	41	2		
34	167	3	46	16	0	18	191	3	38	13	0	2	215	3	30	11	1
34	152	63	1			18	179	51	2			2	205	43	0		
33	169	1	47	16	0	17	193	1	39	14	1	1	217	1	31	11	0
33	154	61	2			17	180	53	0			1	207	41	1		
32	170	3	45	16	1	16	194	3	37	13	0	0	218	3	29	10	10
32	155	63	0			16	182	51	1			0	209	39	2		
31	172	1	46	16	0	15	196	1	38	13	0						
31	157	61	1			15	184	49	2								

Also solved by L. P. Shidy, R. A. Johnson, F. H. Loud, H. N. Carleton, B. F. Yanney.

NOTES AND NEWS.

Edited by E. J. Moulton, Northwestern University, Evanston, Ill.

Mr. Warren Weaver has been appointed instructor in mathematics at the University of Wisconsin.

Mr. P. A. Fraleigh, of Cornell University, has been appointed instructor in mathematics at Dartmouth College.

Professor O. D. Kellogg, of the University of Missouri, has been appointed lecturer in Harvard University for the year 1919–20.